Artificial Intelligence

Exact Inference:
Variable Elimination
Ch. 14.4
A Bayesian network is a compact representation of a joint probability distribution.
Types of Inference Tasks

Compute the probability or most likely state of a set of query variables given observed values of some other variables

- Belief updating
  \[ P(B=t \mid N_c=f) \]

- Most probable explanation (MPE) queries
  \[ \text{argmax}_{B,Z,A} P(B,Z,A \mid N_c=f, Ret) \]

- Maximum A Posteriori (MAP) queries
  \[ \text{argmax}_{B,E} P(B,E \mid N_c=t, Ret) \]
Inference Direction

• Types of inference by reasoning direction

- Diagnostic
- Causal
- (Explaining Away) Intercausal
- Mixed
Examples

- Diagnostic inferences: from effect to causes.
- Causal Inferences: from causes to effects.
- Intercausal Inferences:
- Mixed Inference:
Variable Elimination Algorithm

General idea:
• \( P(Y | E = e) = \alpha P(Y,E = e) = \alpha \Sigma h P(Y,E= e, H = h) \)
• Write query in the form

\[
P(X_n, e) = \sum_{x_k} \cdots \sum_{x_3} \sum_{x_2} \prod_{i} P(x_i | pa_i)
\]

• Pick an elimination order
• For each variable in the ordering:
  – group together relevant terms
  – Perform summation over these terms, getting a new term
  – Insert the new term into the product
Example

• “Asia” network:

- Visit to Asia
- Smoking
- Tuberculosis
- Lung Cancer
- Abnormality in Chest
- Bronchitis
- X-Ray
- Dyspnea
• We want to compute $P(d)$
• Need to eliminate: $v,s,x,t,l,a,b$

Initial factors

$$P(v,s,t,l,a,b,x,d) =$$
$$P(v)P(s)P(t|v)P(l|s)P(b|s)P(a|t,l)P(x|a)P(d|a,b)$$

“Brute force approach”

$$P(d) = \sum_{x} \sum_{b} \sum_{a} \sum_{l} \sum_{t} \sum_{s} \sum_{v} P(v,s,t,l,a,b,x,d)$$

Complexity is exponential in the size of the graph (number of variables) = $T$. $N=$number of states for each variable $O(N^T)$
• We want to compute $P(d)$
• Need to eliminate: $v, s, x, t, l, a, b$

Initial factors

\[
P(v)P(s)P(t \mid v)P(l \mid s)P(b \mid s)P(a \mid t, l)P(x \mid a)P(d \mid a, b)
\]

Eliminate: $v$
Compute:

\[
f_v(t) = \sum_v P(v)P(t \mid v)
\]

\[
\Rightarrow f_v(t)P(s)P(l \mid s)P(b \mid s)P(a \mid t, l)P(x \mid a)P(d \mid a, b)
\]

Note: $f_v(t) = P(t)$
In general, result of elimination is not necessarily a probability term
Potential multiplication and marginalization

\[ f_v(t) = \sum_v P(v)P(t|v) \]

- \[ P(V) = \begin{array}{c|c} v & 0.4 \\ \hline \sim v & 0.6 \end{array} \]

- \[ P(T|V) = \begin{array}{c|c|c} & v & \sim v \\ \hline t & 0.2 & 0.5 \\ \sim t & 0.8 & 0.5 \end{array} \]
• We want to compute $P(d)$
• Need to eliminate: $s, x, t, l, a, b$

• Initial factors

\[
P(v)P(s)P(t \mid v)P(l \mid s)P(b \mid s)P(a \mid t, l)P(x \mid a)P(d \mid a, b)
\]

$\Rightarrow f_v(t)P(s)P(l \mid s)P(b \mid s)P(a \mid t, l)P(x \mid a)P(d \mid a, b)$

Eliminate: $s$

Compute:

\[
f_s(b, l) = \sum_s P(s)P(b \mid s)P(l \mid s)
\]

$\Rightarrow f_v(t)f_s(b, l)P(a \mid t, l)P(x \mid a)P(d \mid a, b)$

Summing on $s$ results in a factor with two arguments $f_s(b, l)$

In general, result of elimination may be a function of several variables
Potential multiplication and marginalization

\[ f_s(b, l) = \sum_s P(s) P(b \mid s) P(l \mid s) \]

- \( P(S) = \)
  
<table>
<thead>
<tr>
<th>s</th>
<th>0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>~s</td>
<td>0.6</td>
</tr>
</tbody>
</table>

- \( P(B \mid S) = \)
  
<table>
<thead>
<tr>
<th>s</th>
<th>\s</th>
<th>~s</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>0.2</td>
<td>0.5</td>
</tr>
<tr>
<td>~b</td>
<td>0.8</td>
<td>0.5</td>
</tr>
</tbody>
</table>

- \( P(L \mid S) = \)
  
<table>
<thead>
<tr>
<th>s</th>
<th>~s</th>
</tr>
</thead>
<tbody>
<tr>
<td>l</td>
<td>0.2</td>
</tr>
<tr>
<td>~l</td>
<td>0.8</td>
</tr>
</tbody>
</table>
Vector representation

\[ f_s(b, l) = \sum_{s} P(s)P(b | s)P(l | s) \]

- \( P(S) = \)
  
  \[
  \begin{array}{c|c}
  s & 0.4 \\
  \sim s & 0.6 \\
  \end{array}
  \]

- \( P(B|S) = \)
  
  \[
  \begin{array}{c|cc}
  & s & \sim s \\
  b & 0.2 & 0.5 \\
  \sim b & 0.8 & 0.5 \\
  \end{array}
  \]

- \( P(L|S) = \)
  
  \[
  \begin{array}{c|cc}
  & s & \sim s \\
  l & 0.2 & 0.5 \\
  \sim l & 0.8 & 0.5 \\
  \end{array}
  \]
• We want to compute \( P(d) \)
• Need to eliminate: \( x, t, l, a, b \)

• Initial factors

\[
P(v)P(s)P(t \mid v)P(l \mid s)P(b \mid s)P(a \mid t, l)P(x \mid a)P(d \mid a, b)
\]
\[
\Rightarrow f_v(t)f_s(b, l)P(a \mid t, l)P(x \mid a)P(d \mid a, b)
\]

Eliminate: \( x \)

Compute:

\[
f_x(a) = \sum_x P(x \mid a)
\]
\[
\Rightarrow f_v(t)f_s(b, l)f_x(a)P(a \mid t, l)P(d \mid a, b)
\]

Note: \( f_x(a) = 1 \) for all values of \( a \) !!
• We want to compute $P(d)$
• Need to eliminate: $t, l, a, b$

• Initial factors

$P(v)P(s)P(t | v)P(l | s)P(b | s)P(a | t, l)P(x | a)P(d | a, b)$
$\Rightarrow f_v(t)P(s)P(l | s)P(b | s)P(a | t, l)P(x | a)P(d | a, b)$
$\Rightarrow f_v(t)f_s(b, l)P(a | t, l)P(x | a)P(d | a, b)$
$\Rightarrow f_v(t)f_s(b, l)f_x(a)P(a | t, l)P(d | a, b)$

Eliminate: $t$

Compute: $f_t(a, l) = \sum_t f_v(t)P(a | t, l)$

$\Rightarrow f_s(b, l)f_x(a)f_t(a, l)P(d | a, b)$
• We want to compute $P(d)$
• Need to eliminate: $l, a, b$

• Initial factors

$$P(v)P(s)P(t | v)P(l | s)P(b | s)P(a | t, l)P(x | a)P(d | a, b)$$
$$\Rightarrow f_v(t)P(s)P(l | s)P(b | s)P(a | t, l)P(x | a)P(d | a, b)$$
$$\Rightarrow f_v(t)f_s(b, l)P(a | t, l)P(x | a)P(d | a, b)$$
$$\Rightarrow f_v(t)f_s(b, l)f_x(a)P(a | t, l)P(d | a, b)$$
$$\Rightarrow \underline{f_s(b, l)}f_x(a)f_t(a, l)P(d | a, b)$$

Eliminate: $l$

Compute: $$f_i(a, b) = \sum_l f_s(b, l)f_t(a, l)$$
$$\Rightarrow f_i(a, b)f_x(a)P(d | a, b)$$
• We want to compute $P(d)$
• Need to eliminate: $b$

• Initial factors

$$P(v)P(s)P(t \mid v)P(l \mid s)P(b \mid s)P(a \mid t,l)P(x \mid a)P(d \mid a,b)$$

$$\Rightarrow f_v(t)P(s)P(l \mid s)P(b \mid s)P(a \mid t,l)P(x \mid a)P(d \mid a,b)$$

$$\Rightarrow f_v(t)f_s(b,l)P(a \mid t,l)P(x \mid a)P(d \mid a,b)$$

$$\Rightarrow f_v(t)f_s(b,l)f_x(a)P(a \mid t,l)P(d \mid a,b)$$

$$\Rightarrow f_s(b,l)f_x(a)f_t(a,l)P(d \mid a,b)$$

$$\Rightarrow f_l(a,b)f_x(a)P(d \mid a,b) \Rightarrow f_a(b,d) \Rightarrow f_b(d)$$

Eliminate: $a,b$

Compute: $f_a(b,d) = \sum_a f_l(a,b)f_x(a)p(d \mid a,b) \quad f_b(d) = \sum_b f_a(b,d)$
• Different elimination ordering:
• Need to eliminate: $a, b, x, t, v, s, l$

• Initial factors

\[
P(v)P(s)P(t \mid v)P(l \mid s)P(b \mid s)P(a \mid t, l)P(x \mid a)P(d \mid a, b)
\]
\[
= g_a(l, t, d, b, x)P(v)P(s)P(t \mid v)P(l \mid s)P(b \mid s)
\]
\[
= g_b(l, t, d, x, s)P(v)P(s)P(t \mid v)P(l \mid s)
\]
\[
= g_x(l, t, d, s)P(v)P(s)P(t \mid v)P(l \mid s)
\]
\[
= g_t(l, t, s, v)P(v)P(s)P(l \mid s)
\]
\[
= g_v(l, d, s)P(s)P(l \mid s)
\]
\[
= g_s(l, d)
\]
\[
= g_l(d)
\]

Complexity is exponential in the size of the factors!
Elimination orders

- Need to eliminate: $a,b,x,t,v,s,l$
- Initial factors

\[
P(v)P(s)P(t|v)P(l|s)P(b|s)P(a|t,l)P(x|a)P(d|a,b)
\]
Elimination orders

• The elimination order has a large effect on the efficiency of VE
• Finding the optimal elimination order is an NP-hard problem
• Typically greedy strategies are used
  – Min-Neighbors: The cost of a node is the number of neighbors it has in the current graph.
  – Min-Weight: The cost of a node is the product of weights --- number of states --- of its neighbors.
  – Min-Fill: The cost of a node is the number of extra edges introduced when eliminating a variable
Example: Min-fill order

\[ P(v)P(s)P(t \mid v)P(l \mid s)P(b \mid s)P(a \mid t,l)P(x \mid a)P(d \mid a,b) \]